Mathematical Induction: History And Applications

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Outline

Introduction

What is the mathematical induction?

Example

Prove that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \quad \forall n \in \mathbb{N}.$$

Proof by Induction:
Base Case: For $n = 1$,

$$1 = \frac{1(1+1)}{2} = 1.$$

So the statement holds.

Introduction

Inductive Step: Assume the statement holds for $n = k$, i.e.

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

For $n = k + 1$,

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Thus, the formula also holds for $n = k + 1$.
Conclusion: By the principle of mathematical induction, the formula holds for all $n \in \mathbb{N}$.

The Inductive Thought in China

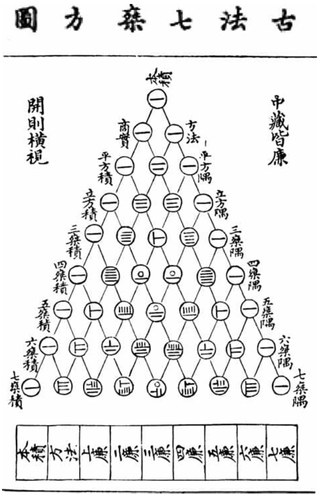


Figure: Yang Hui's Triangle (1261 AD)

Ancient Chinese Terminology in Yang Hui's Diagram

古文字（Classical Chinese）

- 古法七秦方圖
- 開則橫視
- 中藏皆康
- 本積
- 二廉
- 三廉

English Translation

- Ancient Method of the Seven Qins Diagram
- Open and view horizontally
- Within lies harmony
- Root product
- Second coefficient
- Third coefficient

Yang Hui's Triangle (1261 AD)

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

Binomial Theorem

Step 1: Observe coefficients from polynomial expansions

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$n \in \mathbb{N} \cup \{0\}$, $k \in \{0, 1, 2, \ldots, n\}$

Verbal rule (Yang Hui)

“Start from one, place ones at both sides, and each inner number equals the sum of the two above it.”

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

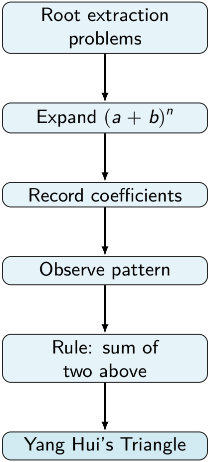
Step 2: Discover the recursive rule

$$T(n,k) = T(n-1,k-1) + T(n-1,k), \qquad T(n,0) = T(n,n) = 1.$$

Discovery of Yang Hui's Triangle

How the rule was found (empirical path):

1. Start with **root extraction** problems.
2. Expand $(a+b)^n$ repeatedly to get coefficients.
3. Record coefficients in calculation tables.
4. Notice: each number $=$ **sum of two above**.
5. Form the "**Chart of Increasing and Multiplying Method**".



Related Works

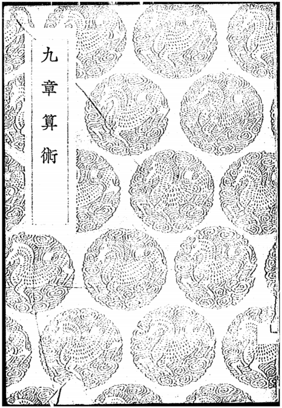


Figure: Jiuzhang Suanshu (c.3 BCE)

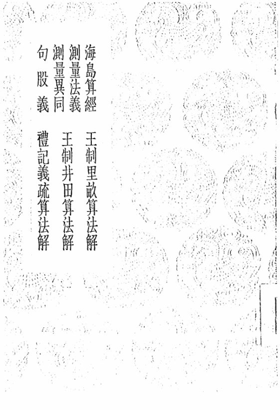


Figure: Haidao Suanjing (ca.263 CE)

Haidao Suanjing (The Sea Island)

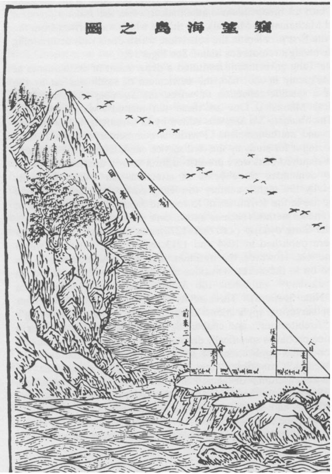


Figure: Illustration of *Haidao Suanjing*

History of *Haidao Suanjing (AD 263)*

Ancient Chinese mathematical work by Liu Hui, continuing *The Nine Chapters on the Mathematical Art*.

- Height and distance measurement
- "Repeated difference" surveying method
- Practical mathematical approach

In-and-Out Complementary Principle (IOCP)

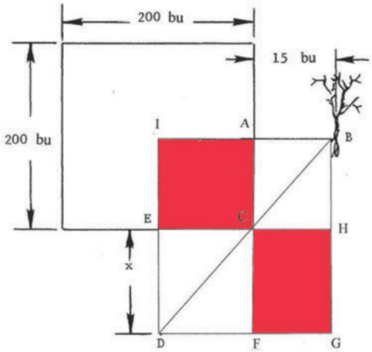


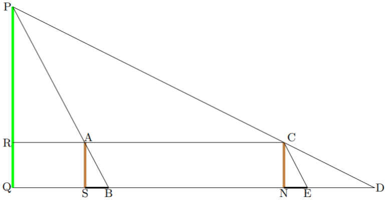
Figure: Geometric reasoning of IOCP (Swetz, 2014)

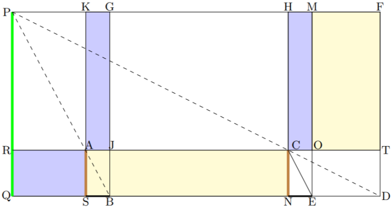
In-and-Out Complementary Principle (IOCP)

- Two inner rectangles represent equal-area relationships.
- Replaces similarity with area equivalence:

$$IE \times EC = CF \times FG$$

The Sea Island Problem



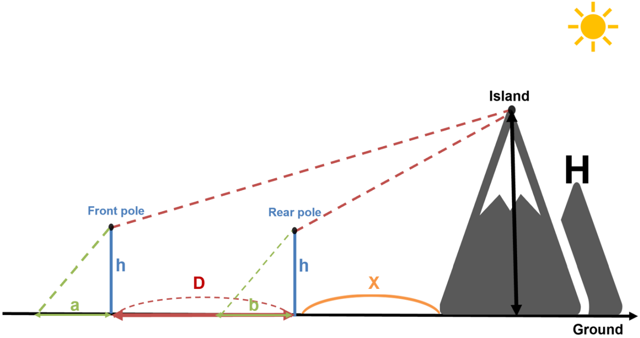


$$\overline{PQ} = \frac{\overline{AS} \cdot \overline{SN}}{\overline{ND} - \overline{SB}} + \overline{AS} \quad (1)$$

$$\overline{QS} = \frac{\overline{SB} \cdot \overline{SN}}{\overline{ND} - \overline{SB}} \quad (2)$$

Figure explanation:
Swetz, F. (1992). *The Sea Island Mathematical Manual*. Penn State Press. The diagrams illustrate the geometric method used to measure the height and distance of the island across the sea.

"Repeated Difference" Method



"Repeated Difference" Method: Measure an island using two poles and proportion.

Example: Using the general formulas:

$$\mathrm{H} = h + \frac{hD}{b-a}, \quad \mathrm{x} = \frac{aD}{b-a}$$

Given parameters: h = 3 m, D = 1000 m, a = 123 m, b = 127 m
Then: b - a = 127 - 123 = 4 m

$$\mathrm{H} = 3 + \frac{3 \times 1000}{4} = 3 + 750 = 753m$$

$$\mathrm{x} = \frac{123 \times 1000}{4} = 30{,}750m$$

Summary

From specific to general:

- *Example 1:* Measuring a sea island
- *Example 2:* Applying the same method to a mountain
- *Example 3:* Extending the method to towers and city walls

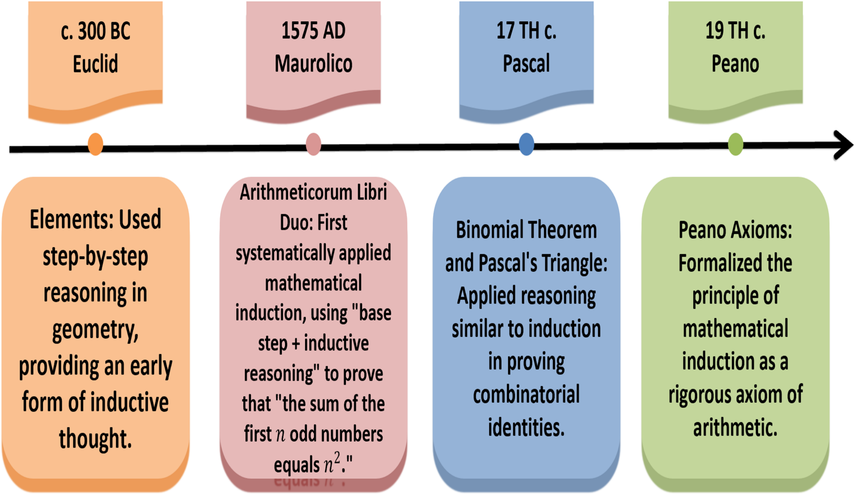
Inductive idea:

- Ancient mathematicians derived general principles through repeated measurement problems
- Demonstrates reasoning "from examples to general law," similar in spirit to mathematical induction

Key Features of Chinese Induction

1. **Practice-oriented**: Focused on patterns in calculations, e.g. Yang Hui's Triangle.
2. **Implicit Induction**: No formal definition, but inductive reasoning applied in practice.
3. **Classical Works**:
   - *Jiuzhang Suanshu*: Systematic problem-solving.
   - *Haidao Suanjing*: Geometrical calculations with inductive patterns.
4. **Contribution**: Provided early forms of inductive reasoning, influencing later mathematical developments.

The Development In the West — Time Table



Ancient Greece

- Strong emphasis on logical reasoning
- Relied mainly on deduction rather than induction.

Euclid's Elements

Publication: c.300 BC
Pages: 13 books
Including: definitions, postulates, geometric constructions, and theorems with proofs.
Core Achievements: the Pythagorean theorem, Thales' theorem, the Euclidean algorithm, the theorem of infinitely many primes, and the construction of regular polygons/polyhedra.

Feature

Focused on building a logical system,
but without explicit mathematical inductive reasoning.

Euclid (fl.c.300 BC)

Example:
Geometric progression: $1, 2, 4$ (ratio 2, starting from unit).
Sum: $1 + 2 + 4 = 7$ (prime).
Product: $7 \times 4 = 28$
Perfect number, since
$1 + 2 + 4 + 7 + 14 = 28$
$1 + 2 + 3 = 6$
$1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 = 496$

at **Base Case:**
Construct $1, 2, 4$, sum as prime 7.

bt **Recursive Construction:**
Use geometric progression properties (Prop. 7.19–20) to deduce 28's divisor sum.

Francesco Maurolico (1494–1575)

Arithmeticorum Libri Duo (1575)

Francesco Maurolico proved that: The sum of consecutive odd numbers forms a perfect square.

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Examples

- When $n = 1$: $1 = 1^2 = 1$
- When $n = 2$: $1 + 3 = 4 = 2^2$
- When $n = 3$: $1 + 3 + 5 = 9 = 3^2$
- When $n = 4$: $1 + 3 + 5 + 7 = 16 = 4^2$
- When $n = 5$: $1 + 3 + 5 + 7 + 9 = 25 = 5^2$

Mathematical Induction

We prove the identity:

$$P(n) : 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Base case: $n = 1$

$$1 = 1^2 \qquad \text{True}$$

Inductive step: Assume the statement holds for some $n = k$, i.e.,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

We must show it holds for $n = k + 1$:

$$\begin{aligned} &1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] \\ &= k^2 + (2k + 1) \quad \text{(by the induction hypothesis)} \\ &= (k + 1)^2 \end{aligned}$$

Conclusion: By mathematical induction, the statement *P(n)* holds for all $n \in \mathbb{N}$.

Blaise Pascal (1623 - 1662)

Figure: Pascal's Arithmetical Triangle

	1	2	3	4	5	6	7	8	9	10
1	G 1	σ 1	π 1	λ 1	μ 1	δ 1	ζ 1	1	1	1
2	φ 1	ψ 2	θ 3	R 4	S 5	N 6	7	8	9	
3	A 1	B 3	C 6	ω 10	ξ 15	21	28	36		
4	D 1	E 4	F 10	ρ 20	Y 35	56	84			
5	H 1	M 5	K 15	35	70	126				
6	P 1	Q 6	21	56	126					
7	V 1	7	28	84						
8	1	8	36							
9	1	9								
10	1									

Key Work

Traité du triangle arithmétique
(Written in 1654, officially published in 1665)

35+21=56,
1+4+10+20+35=70,
1+6+21+56=84,
1+1+1+1+1+2+3+4
= 14+1 = 15

Pascal and Inductive Thinking

- Studied number patterns and the Binomial Theorem using recursion.
- Pascal's Triangle (similar to Yang Hui's Triangle) reflects inductive ideas.

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$n \in \mathbb{N} \cup \{0\}$, $k \in \{0, 1, 2, \ldots, n\}$

Proof of Pascal's Rule (1/3)

Statement of Pascal's Rule

For integers $n \geq 1$ and $1 \leq k \leq n$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Proof by Mathematical Induction

Base Case: For $n = 1$,

$$\binom{2}{1} = 2, \binom{1}{1} + \binom{1}{0} = 1 + 1 = 2$$

$$2 = 2$$

Hence, the base case holds.

Proof of Pascal's Rule (2/3)

Proof by mathematical Inductive Step (continue)

Assume for some $n \geq 1$, the identity holds:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

We need to show it holds for $n+1$:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Proof of Pascal's Rule (3/3)

By Mathematical Induction

Using the factorial definition of binomial coefficients.

$$\begin{aligned}
\binom{n+1}{k} &= \frac{(n+1)!}{k!(n+1-k)!} \\
&= \frac{(n+1)\cdot n!}{k\cdot(k-1)!\cdot(n+1-k)!} = \frac{[k+(n+1-k)]\cdot n!}{k\cdot(k-1)!\cdot(n+1-k)!} \\
&= \frac{k\cdot n!}{k\cdot(k-1)!\cdot(n+1-k)!} + \frac{(n+1-k)\cdot n!}{k\cdot(k-1)!\cdot(n+1-k)!} \\
&= \frac{n!}{(k-1)!(n+1-k)!} + \frac{n!}{k!(n-k)!}
\end{aligned}$$

Thus, By mathematical induction Pascal's Rule holds for all $n \geq 1$.

Proof of Binomial Theorem (1/3)

Statement

For any integer $n \geq 0$,

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k.$$

Base Case

For $n = 0$, $(1+z)^0 = 1$, and

$$\sum_{k=0}^{0} \binom{0}{k} z^k = \binom{0}{0} z^0 = 1.$$

Hence,the base case holds.

Proof of Binomial Theorem (2/3)

Induction Step

Assume $(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$ holds.
Then,

$$(1+z)^{n+1} = (1+z)(1+z)^n = (1+z)\sum_{k=0}^{n} \binom{n}{k} z^k.$$

Expanding,

$$(1+z)^{n+1} = \sum_{k=0}^{n} \binom{n}{k} z^k + \sum_{k=0}^{n} \binom{n}{k} z^{k+1}.$$

Re-index the second sum by letting $m = k+1$:

$$\sum_{m=1}^{n+1} \binom{n}{m-1} z^m.$$

Proof of Binomial Theorem (3/3)

By mathematical induction

Combine both sums:

$$(1+z)^{n+1} = \sum_{m=0}^{n+1} \left[\binom{n}{m} + \binom{n}{m-1} \right] z^m.$$

By Pascal's Rule,

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}.$$

Thus,

$$(1+z)^{n+1} = \sum_{m=0}^{n+1} \binom{n+1}{m} z^m.$$

Therefore, by mathematical induction the binomial theorem holds for all $n \geq 0$.

Relationship Summary

Connection between Pascal's Rule and the Binomial Theorem

- **Pascal's Rule:** Defines the recursive relation of coefficients

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

- **Binomial Theorem:** Uses this recursion to prove inductively that

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k.$$

- **Order:**

Pascal's Rule $\Rightarrow$ Binomial Theorem.

Fibonacci Sequence (1202) : Rabbit Breeding Example

- **Assumptions**: 1 initial newborn pair; matures in 1 month; adults breed 1 new pair monthly; no death.
- **Total rabbit pairs (vs. Fibonacci terms)**:

Figure: A special stamp

Month	Total Pairs	Fibonacci Term
0	1	F_0
1	1	F_1
2	2	F_2
3	3	F_3
4	5	F_4
5	8	F_5

Figure: Newborn rabbit pair statistics

$$\boxed{F_n = F_{n-1} + F_{n-2}} \quad (F_0 = 1,\ F_1 = 1)$$

Fibonacci Numbers in Pascal's Triangle

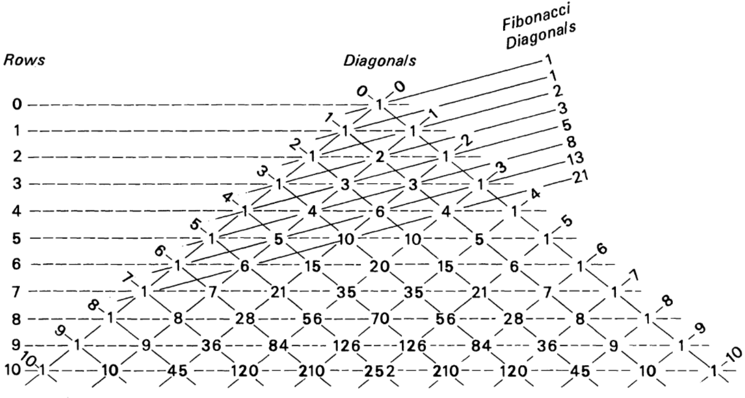


Figure: Pascal's Arithmetical Triangle.
(shallow diagonal is a Fibonacci number)

Observation: Fibonacci Numbers from Pascal's Triangle

Idea

By summing the numbers along the *shallow diagonals* of Pascal's Triangle, we obtain the Fibonacci sequence:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ \ldots$$

Diagonal k	0	1	2	3	4	5
Sum F_k	1	1	2	3	5	8

Combinatorial Form and Recurrence Relation

Definition

The n-th Fibonacci number can be expressed using binomial coefficients:

$$F_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}.$$

$\lfloor n/2 \rfloor$: the floor function of $n/2$ (the greatest integer less than or equal to $n/2$), serving as the upper limit of the summation.

Examples:

$$F_5 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 1 + 4 + 3 = 8,$$

$$F_6 = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = 1 + 5 + 6 + 1 = 13.$$

Inductive Proof of $F_n = F_{n-1} + F_{n-2}$

Step 1. Pascal's Identity

$$\binom{n-i}{i} = \binom{n-1-i}{i} + \binom{n-1-i}{i-1}.$$

Step 2. Substitute into the Definition of F_n

$$\begin{aligned} F_n &= \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \left[\binom{n-1-i}{i} + \binom{n-1-i}{i-1} \right]. \end{aligned}$$

Inductive Proof of $F_n = F_{n-1} + F_{n-2}$

Step 3. Rearrange the Two Sums

$$F_n = \underbrace{\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-1-i}{i}}_{F_{n-1}} + \underbrace{\sum_{j=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-2-j}{j}}_{F_{n-2}}.$$

$$\boxed{F_n = F_{n-1} + F_{n-2}}$$

Conclusion

Therefore, by mathematical induction via Pascal's Identity and sum rearrangement, $F_n = F_{n-1} + F_{n-2}$, proving the Fibonacci recurrence.

Fibonacci Number Summary

Summary

- Fibonacci numbers naturally arise from Pascal's Triangle through **diagonal summation**.
- The recurrence relation $F_n = F_{n-1} + F_{n-2}$ follows from **Pascal's identity**.
- This reveals a deep link between **combinatorics** and **number theory**.

Giuseppe Peano and the Principle of Induction

- In the 19th century, Peano formalized induction within his axioms of natural numbers.
- Mathematical induction became a rigorous logical proof method.

Principle of Induction

1. $P(n)$ represents a statement or property about the natural number n.
2. $P(1)$ is true.
3. If $P(k)$ is true, then $P(k+1)$ is true.
4. Therefore, $P(n)$ is true for all $n \in \mathbb{N}$.

Giuseppe Peano (1858 - 1932)

What is Strong Induction?

- Core : To prove $P(n)$, assume $P(1), P(2), \ldots, P(k)$ all hold (not just $P(k)$), then derive $P(k+1)$.
- Use case : Propositions dependent on multiple previous terms (e.g., Fibonacci sequence).

Peano's Core Contributions to Induction

- Turned induction from an *intuitive method* into a *logically rigorous tool* via his inductive axiom (5th Peano axiom).
- Established induction as a fundamental part of natural number theory—no longer just a "trick" for proofs.

Key Features of Western Induction

1. **Theoretical**: From experience to abstraction.
2. **Systematic**: Based on an axiomatic framework.
3. **Logical**: A rigorous proof technique in modern mathematics.

Key Differences

Aspect	Chinese Tradition (Liu Hui, Yang Hui)	European Tradition (Induction)
Nature	Visual, geometric, and constructive.	Formal, logical, and axiomatic.
Focus	Practical results, algorithms, and applications.	Abstract theory, deduction from axioms.
Proof Style	"Proof without words," geometric rearrangements.	Base case and inductive step.
Example	Piling up squares to prove summation formulas.	Proving summation formulas algebraically.

Table: Comparison of Chinese and European Proof Styles

Applications

- Computer Science
- Combinatorics
- Geometry and Graph Theory
- Mathematical Analysis
- Logic and Formal Systems
- Applied Mathematics and Engineering

The End

THANK YOU

Special thanks to Dr. R. L. Herman

Reference

- Wikipedia. (n.d.). Euclid's Elements. Retrieved from `https://en.wikipedia.org/wiki/Euclid%27s_Elements`
- Cajori, F. (1918). Origin of the Name "Mathematical Induction." The American Mathematical Monthly, 25(5), 197–201. https://doi.org/10.2307/2972638
- J. W. A. Young. (1908). The American Mathematical Monthly, Vol. 15, No. 8/9 (Aug. - Sep., 1908), pp. 145-153
- Swetz, F. (1979). The Evolution of Mathematics in Ancient China. Mathematics Magazine, 52(1), 10–19. https://doi.org/10.2307/2689967
- Frank J. Swetz. (2012). Mathematics Magazine, Vol. 85, No. 1 (February 2012), pp. 3-11. Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America
- Frank J. Swetz. "Led Astray by a Right Triangle: Misconception, Epiphany, and Redemption - The Epiphany"